ECE 388 – Automatic Control

Controllability and State Feedback Control

Associate Prof. Dr. Klaus Schmidt

Department of Mechatronics Engineeering – Çankaya University

Compulsory Course in Electronic and Communication Engineering
Credits (2/2/3)

Course Webpage: http://ECE388.cankaya.edu.tr

Controllability: Preliminaries

State-space Model

$$\dot{x}(t) = A\,x(t) + b\,u(t)$$
$$y(t) = c^T\,x(t) + d_d\,u(t)$$

Transfer Function

$$G(s) = c_d^T(sI - A)^{-1}b + d$$

Solution of the State Equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

Example

Gap 1

Controllability: Definition

Controllability Definition

A linear system is called completely controllable if it is possible for each pair of states x_0, x_1 to find a control input $u(t)$ that moves the system state from x_0 to x_1 in a specified transfer time t_T

Controllability Test by Kalman

A linear system of order n is completely controllable if and only if the controllability matrix

$$\mathcal{C} = \begin{bmatrix} b & A\,b & \cdots & A^{n-2}\,b & A^{n-1}\,b \end{bmatrix}$$

has full rank n (n is the order of the state space model)

⇒ Check the rank of $\mathcal{C}$ to verify controllability

Controllability Stability State Feedback Control

Controllability: Example

Computation

Gap 2

Klaus Schmidt Department
ECE 388 – Automatic Control

State Feedback Control: Alternative Controllability Test

Hautus Test

An eigenvalue λ of A is controllable (uncontrollable) if and only if the matrix $\begin{bmatrix}(\lambda I - A) & b\end{bmatrix}$ has (does not have) full rank n

$\rightarrow$ System is controllable if all eigenvalues are controllable

Example

Gap 3

Controllability Stability State Feedback Control

Stability: State Space Models

Definition

A linear system with the dynamic matrix A is asymptotically stable if all eigenvalues of A lie in the open left half plane

$\Rightarrow$ Stronger condition than BIBO stability: $G(s) = c^T(sI - A)^{-1}b + d$ can be BIBO stable even if A has eigenvalues in the right half plane!

Example

Gap 4

Klaus Schmidt Department
ECE 388 – Automatic Control

Stability: Example

Computation

Gap 5

State Feedback Control: Idea

Given

$$\dot{x}(t) = A\,x(t) + b\,u(t)$$
$$y(t) = c^T\,x(t) + d\,u(t)$$

Goal

- Use feedback of the state vector x to move the eigenvalues of the closed-loop system to desired locations

State Feedback

$$u(t) = k^T\,x(t) + M\,r(t)$$

Design Parameters

- Feedback vector k; pre-filter M

State Feedback Control: Block Diagram

Illustration

Gap 6

Practical Fact

- All state variables must be measurable

State Feedback Control: Closed Loop

Computation

Gap 7

State Space Model

$$\dot{x}(t) = \underbrace{(A + b\,k^T)}_{\tilde{A}}\, x(t) + \underbrace{b\,M}_{\tilde{b}}\, r(t)$$

$$y(t) = \underbrace{(c^T + d\,k^T)}_{\tilde{c}^T}\, x(t) + \underbrace{d\,M}_{\tilde{d}}\, r(t)$$

Notation in Closed Loop

- Dynamic matrix
  $\tilde{A} = A + b\,k^T$
- Transfer function
  $\tilde{G}(s) = \tilde{c}^T(sI - \tilde{A})^{-1}\tilde{b} + \tilde{d}$

State Feedback Control: Closed Loop

Closed-loop Requirements

- Stability
  → All eigenvalues of $\tilde{A}$ should lie in the OLHP
- Sufficient performance
  → Suitable choice of the closed-loop eigenvalues (for example far away enough from the imaginary axis) using design parameter k
- Zero steady-state error
  → Suitable choice of the design parameter M

Questions

- When is it possible to assign the poles of the closed loop?
- How can we compute the design parameters k and M?

Controllability Stability State Feedback Control

State Feedback Control: Pole Assignment

Choice of the Pole Locations

If a linear system is completely controllable, then the eigenvalues of the closed-loop dynamic matrix $\tilde{A} = A + b\,k^T$ can be assigned arbitrarily by a suitable choice of k

Pole Assignment for Complete Controllability

- System order n: Choice of the closed loop characteristic polynomial (for example using desired eigenvalue locations)

$$p(s) = p_0 + p_1 s + \cdots + p_{n-1}s^{n-1} + s^n$$

  → We want that $p(s) = \det(sI - A - b\,k^T)$

Klaus Schmidt Department
ECE 388 – Automatic Control

State Feedback Control: Pole Assignment

Pole Assignment for Complete Controllability

- Formula of Ackermann: Compute the vector v such that $v^T = \begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix} \mathcal{C}^{-1}$
- Compute state feedback vector k using $p(s)$ and v

$$k^T = -p_0 v^T - p_1 v^T A - \cdots - p_{n-1} v^T A^{n-1} - v^T A^n$$

Example

Gap 8

State Feedback Control: Example

Computation

Gap 9

State Feedback Control: Uncontrollable Eigenvalues

Hautus Test

- Find uncontrollable eigenvalues
- All uncontrollable eigenvalues must also appear in the closed loop

Pole Assignment by Comparison of Coefficients

- Determine the characteristic polynomial of the closed loop for $k^T = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$ ($k_1, \ldots, k_n$ are free parameters)

  $\rightarrow \det(sI - A - b\,k^T)$
- Choose a desired closed loop characteristic polynomial $p(s)$ that preserves uncontrollable eigenvalues

  $\rightarrow$ Evaluate $\det(sI - A - b\,k^T) = p_0 + p_1 s + \cdots p_{n-1}s^{n-1} + p_n s^n$
- Compute the free parameters $k_1, k_2, \ldots, k_n$ by comparison of coefficients

State Feedback Control: Example

Computation

Gap 10

State Feedback Control: Example

Computation

Gap 11

State Feedback Control: Pre-Filter

Goal

- For unit reference step, we want to reach steady-state output value 1

Solution

- Apply final value theorem to the closed-loop transfer function:

$$
\begin{aligned}
1 &= \lim_{t\to\infty} y(t) = \lim_{s\to 0} \tilde{G}(s) \\
&= \lim_{s\to 0} \big((c^T + d\,k^T)(sI - A - b\,k^T)^{-1} b\,M + d\,M\big) \\
&= \big((c^T + d\,k^T)(-A - b\,k^T)^{-1} b + d\big) M \\
&\Rightarrow M = \frac{1}{(c^T + d\,k^T)(-A - b\,k^T)^{-1} b + d}
\end{aligned}
$$

→ Note: eigenvalues of $-A - b\,k^T$ are non-zero

State Feedback Control: Example

Computation

Gap 12