ECE 388 – Automatic Control

Transfer Functions – Step Response

Associate Prof. Dr. Klaus Schmidt

Department of Mechatronics Engineeering – Çankaya University

Compulsory Course in Electronic and Communication Engineering
Credits (2/2/3)

Course Webpage: http://ECE388.cankaya.edu.tr

State Space Model Solution: Description

State Equations

$$\dot{x}(t) = A\,x(t) + b\,u(t), \quad x(0) = x_0$$
$$y(t) = c^T\,x(t) + d\,u(t)$$

Goal

- Determine evolution of state signal x and output signal y depending on initial condition x_0 and input signal u
  $\Rightarrow$ Solve the state space model equation

Matrix Exponential

$$e^{A\,t} = \sum_{k=0}^{\infty} \frac{A^k\,t^k}{k!} = I + A\,t + \frac{A^2\,t^2}{2!} + \frac{A^3\,t^3}{3!} + \cdots$$

Solution of the State Equation: Matrix Exponential

Example

Gap 1

State Space Model Solution: Result

Solution for the State Signal

$$x(t) = e^{A\,t}\,x_0 + \int_0^t e^{A\,(t-\tau)}\,b\,u(\tau)d\tau$$

Solution for the Output Signal

$$y(t) = c^T\,x(t) + d\,u(t) = c^T\,e^{A\,t}\,x_0 + c^T\,\int_0^t e^{A\,(t-\tau)}\,b\,u(\tau)d\tau + d\,u(t)$$

Parts of the Solution

- Zero-input solution ($u \equiv 0$): $y(t) = c^T\,e^{A\,t}\,x_0$
  $\rightarrow$ Solution of the state equation if no input is applied
- Zero-state solution ($x_0 = 0$): $y(t) = c^T\,\int_0^t e^{A\,(t-\tau)}\,b\,u(\tau)d\tau + d\,u(t)$
  $\rightarrow$ Solution of the state equation if the initial condition is zero

Solution of the State Equation: Example

Computation

Gap 2

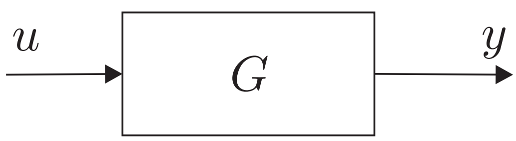
Transfer Functions: Transfer Block Representation

Goal

- Input signal in the Laplace domain: $U(s) \quad \bullet\!\!-\!\!\!-\!\!\circ \quad u(t)$
- Output signal in the Laplace domain: $Y(s) \quad \bullet\!\!-\!\!\!-\!\!\circ \quad y(t)$
- Representation of the input/output relation in the Laplace domain
  $\Rightarrow$ Transfer function $G(s) = \dfrac{Y(s)}{U(s)}$

Transfer Block Representation

- Input/output variables u, y
- Transfer function G



Evaluation of the Transfer Function

- We use the zero-state solution of the state-space model

Transfer Functions: Computation from State Space Model

Linear State Space Model

$$\dot{x} = A \cdot x + b \cdot u$$
$$y = c^T \cdot x + d \cdot u$$

Laplace Transformation

$$sX(s) - x(0) = A \cdot X(s) + b \cdot U(s)$$
$$Y(s) = c^T \cdot X(s) + d \cdot U(s)$$

Computation

Gap 3

Transfer Function

$$G(s) = c^T(sI - A)^{-1}b + d$$

Transfer Functions: Examples

RLC-Circuit

Gap 4

Transfer Functions: Facts

Output Computation in Time Domain

- Convolution: $g(t) \star u(t) = \int_{-\infty}^{\infty} g(t-\tau)\, u(\tau)\, d\tau$

$$\Rightarrow y(t) = g(t) \star u(t)$$

Output Computation Using Transfer Function

- Output in Laplace Domain: $Y(s) = G(s)\, U(s)$
- Inverse Laplace transform: $y(t) = \mathcal{L}^{-1}(Y(s))$

$$\Rightarrow y(t) = g(t) \star u(t) = \mathcal{L}^{-1}(G(s)\, U(s)) = \mathcal{L}^{-1}(Y(s))$$

Examples

Gap 5

Transfer Functions: Output Response

Examples

Gap 6

Transfer Functions: Output Response

Examples

Gap 7

Step Responses: Computation

Definition

The unit step response is the output response of a dynamic system to the Heaviside step function applied at its input

- Heaviside step function: $\sigma(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$

Step Response Computation

- Input: $u(t) = \sigma(t) \circ\!\!-\!\!\bullet\ U(s) = \frac{1}{s}$
- Output: $Y(s) = G(s)U(s) = G(s)\frac{1}{s}\ \bullet\!\!-\!\!\circ\ y(t) = \mathcal{L}^{-1}\big(G(s)\frac{1}{s}\big)$

Gap 8

Step Responses: Integrator Example

Integrator

- State space model: $\dot{x} = u$, $y = x$
- Transfer function: $G(s) = \frac{1}{s}$
- Step response: $Y(s) = G(s)\frac{1}{s} = \frac{1}{s^2}$ $\bullet\!\!-\!\!\circ$ $\sigma(t) \cdot t$

Gap 9

Step Responses: First-order Lag Example

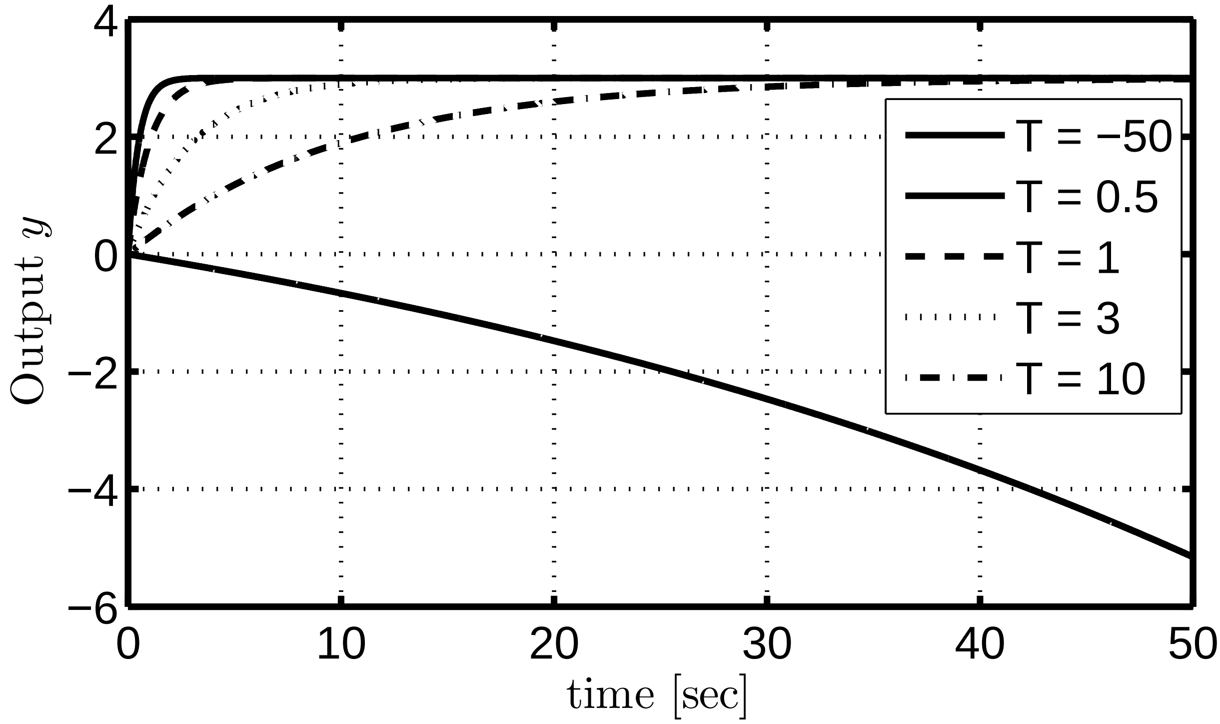
First-order Lag

- State-space model: $\dot{x} = -\frac{1}{T}x + \frac{K}{T}u$, $y = x$
- Transfer function: $G(s) = \frac{K}{1 + s \cdot T}$
- Step response: $Y(s) = \frac{G(s)}{s} = \frac{K}{(1 + T \cdot s)s}$ $\bullet\!\!-\!\!\circ$ $\sigma(t)K(1 - e^{-t/T})$

Gap 10

Step Responses: First-order Lag Example

First-order Lag for Different T



Step Responses: Second-order Lag Example

Second-order Lag

- Model: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -\omega_n^2\\ 1 & -2\,D\,\omega_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ K\end{bmatrix} u;\ y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$
- Transfer function: $G(s) = \dfrac{K\,\omega_n^2}{\omega_n^2 + 2\,D\,\omega_n\,s + s^2}$
- Step response: Different solutions depending on damping constant D

Denominator Zeros

Gap 11

Step Responses: Second-order Lag Example

Case 1: $D \geq 1$ (aperiodic case)

- $G(s) = \dfrac{K\,\omega_n^2}{(s - s_1)(s - s_2)}$
- $y(t) = \sigma(t)\,K\,(1 + \frac{s_1}{s_2 - s_1} e^{s_1\,t} - \frac{s_1}{s_2 - s_1} e^{s_2\,t})$

$\Rightarrow$ Second-order lag can be considered as two first-order lags
$\Rightarrow$ Note: If the real part of s_1 or s_2 is positive, $y(t)$ goes to infinity

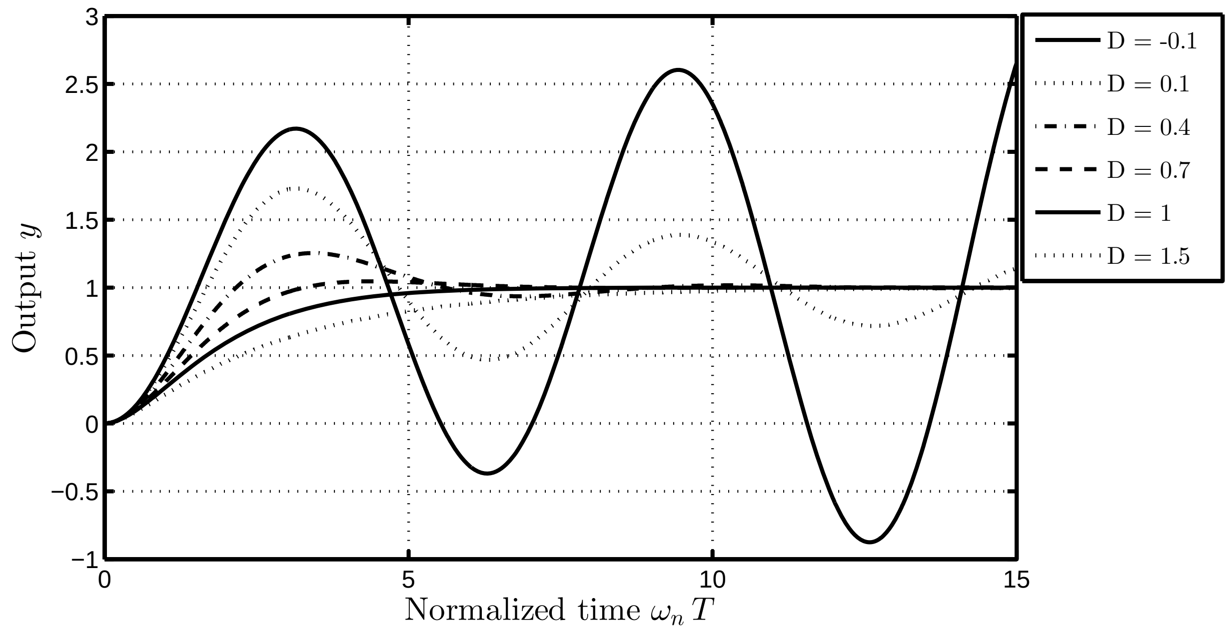
Case 2: $D < 1$ (periodic case)

- $G(s) = \dfrac{K\,\omega_n^2}{(\omega_n^2 + 2\,D\,\omega_n\,s + s^2)}$
- $y(t) = \sigma(t)K\big(1 - \frac{e^{-D\,\omega_n\,t}}{\sqrt{1-D^2}} \sin(\sqrt{1 - D^2}\,\omega_n\,t + \arctan\frac{\sqrt{1-D^2}}{D})\big)$

$\Rightarrow$ The step response shows oscillations
$\Rightarrow$ If $D < 0$, $y(t)$ goes to infinity

Step Responses: Second-order Lag Example

Second-order Lag for Different D



Step Responses: Standardized Transfer Blocks

Proportional Gain

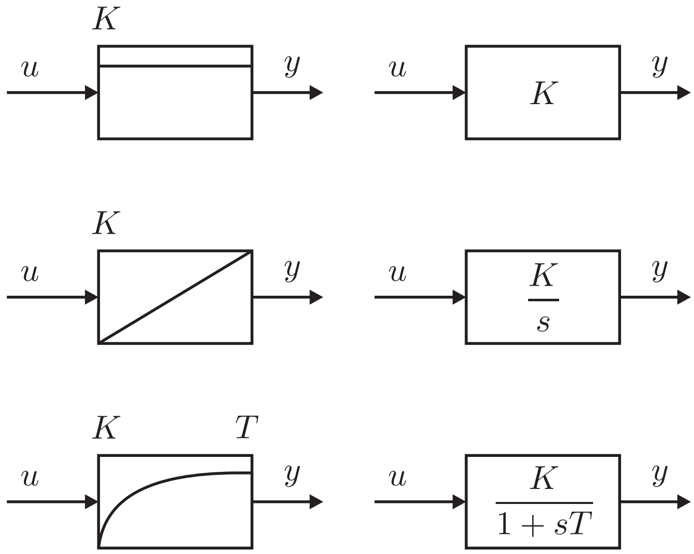
$$y(t) = K\sigma(t)$$

Integrator

$$y(t) = K\sigma(t)t$$

First-order Lag

$$y(t) = K\sigma(t)(1 - e^{-t/T})$$



Laplace Transform: Basic Facts

Definition

- Continuous-time signal $y(t)$, $0 \leq t < \infty$
- (unilateral) Laplace transform: $\mathcal{L}(y(t)) = \int_{0-}^{\infty} y(t)e^{-st}dt$

  $\rightarrow$ we usually write $y(t) \circ\!\!-\!\!\bullet\; Y(s) = \mathcal{L}(y(t))$

Basic Functions

$y(t)$	$\delta(t)$	$\sigma(t)$	t	e^{-at}	$\sin(\omega t)$	$\cos(\omega t)$	$e^{-at}\sin(\omega t)$
$\mathcal{L}(y(t))$	1	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{1}{s+a}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{s}{s^2+\omega^2}$	$\frac{\omega}{(s+a)^2+\omega^2}$