ECE 388 – Automatic Control

Root Locus Plot and Root Locus Design

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Compulsory Course in Electronic and Communication Engineering
Credits (2/2/3)

Course Webpage: http://ECE388.cankaya.edu.tr

Motivation: Task

Reminder

- Basic feedback loop

Gap 1

- Closed loop poles are zeros of $1 + C(s)\,G(s)$

Goal

- Determine how the poles (roots) of the feedback loop change depending on $C(s)$
- Assume that $C(s)$ is given as $C(s) = K\,C'(s)$ with a free gain parameter K

Gap 2

Motivation: Example

Computation

Gap 3

Root Locus Construction Rules: Notation

Open Loop Transfer Function

$$G_{\mathrm{o}}(s) = K\, C'(s)\, G(s) = K\, \frac{N(s)}{D(s)}$$

Pole-Zero Representation of $G_{\mathrm{o}}(s)$

$$G_{\mathrm{o}}(s) = K\, \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

- m zeros: $z_1, \ldots, z_m$
- n poles: $p_1, \ldots, p_n$
- Gain parameter $K > 0$

Root Locus Construction Rules: R1 to R3

R1

- The root locus has $\max(n, m)$ branches

R2

- The root locus starts ($K = 0$) at the poles of $G_o(s)$; there are n zeros of $D(s)$ and $m - n$ poles at $|s| = \infty$ if $m - n > 0$.
- The root locus ends ($K \to \infty$) at the zeros of $G_o(s)$; there are m zeros of $N(s)$ and $n - m$ zeros at $|s| = \infty$ if $n - m > 0$.

R3

- The root locus stays on the real axis on the left of an odd number of poles and zeros of $G_o(s)$

Root Locus Construction Rules: R4 to R5

R4

- The root locus has $n - m$ asymptotes for $|s| \to \infty$ if $n - m > 0$:
  - Intersection of the asymptotes with the real axis at $\sigma = \dfrac{(p_1 + \cdots + p_n) - (z_1 + \cdots + z_m)}{n - m}$
  - Direction of the asymptotes at angles: $\theta = \dfrac{\pi}{n-m}(2\,k + 1)$, $k \in \mathbb{Z}$

R5

- The root locus breaks away from the real axis/joins the real axis for values of s that fulfill

$$N(s)\frac{d}{ds}D(s) - D(s)\frac{d}{ds}N(s) = 0$$

Root Locus Construction Rules: R6 to R8

R6

- The angle of departure from any complex pole p_j is

$$\pi + \angle(p_j - z_1) + \cdots + \angle(p_j - z_m) - \angle(p_j - p_1) - \cdots - \angle(p_j - p_n)$$

R7

- The angle of departure from any complex zero z_j is

$$\pi - \angle(z_j - z_1) - \cdots - \angle(z_j - z_m) + \angle(z_j - p_1) + \cdots + \angle(z_j - p_n)$$

R8

- For intersections of the root locus with the imaginary axis, it holds that $D(s) + K\,N(s)$ is divided by $s^2 + \omega^2$ for some K and ω

Root Locus Construction Rules: General Remarks

Application of the Rules

- In general, not all rules need to be used (some rules might not be applicable)
- The root locus plot gives information about the poles of the closed loop
- The root locus plot is constructed using the open-loop transfer function $G_{\mathrm{o}}(s)$
- For a root $s^\star$ on the root locus, the corresponding value K is computed as

$$K = -\frac{D(s^\star)}{N(s^\star)}$$

- The construction of the root locus plot is formulated for the gain K as free parameter but any free parameter could be used

Examples: $G_o(s) = K\,\frac{s+5}{(s+7)\,(s-2)}$

Computation

Gap 4

Motivation Root Locus Construction Rules Examples Root Locus Design

Examples: $G_o(s) = K\,\frac{s+5}{(s+7)\,(s-2)}$

Computation

Gap 5

Examples: $G_o(s) = K \frac{s+3}{s\,(s^2+2\,s+5)}$

Computation

Gap 6

Examples: $G_o(s) = K \frac{s+3}{s\,(s^2+2\,s+5)}$

Computation

Gap 7

Examples: $G_o(s) = K\,\frac{s+1}{s^2\,(s+9)}$

Computation

Gap 8

Examples: $G_o(s) = K\,\frac{s+1}{s^2\,(s+9)}$

Computation

Gap 9

Root Locus Design: Description

Given Open Loop Transfer Function $G_o(s)$

- Construct root locus plot of $G_o(s)$
- Choose *desirable* closed-loop pole locations
- Compute K from desirable pole locations

Given Plant Transfer Function $G(s)$

- Construct root locus plot of $G(s)$
- Choose *desirable* closed-loop pole locations
- Choose controller poles/zeros such that root locus plot fulfills desirable properties
- Compute K from desirable pole locations

Root Locus Design: Location in the Complex Plane

General Rule for Second-order Lags

- For given t_s (2%) $\Rightarrow Re(p_{1/2}) < -4/t_s$
- For given $D < 1 \Rightarrow \beta := \pi - |\angle(p_{1/2})| < \arctan \sqrt{1-D^2}/D$

Illustration

Gap 10

Root Locus Design: Dominant Complex Poles

Possible Performance Specification

- Damping D
- Settling time $t_s = \dfrac{3}{D\,\omega_n}$ (5%) or $t_s = \dfrac{4}{D\,\omega_n}$ (2%)

  $\Rightarrow$ Determine desired closed-loop pole locations

Illustration

Gap 11

Root Locus Design: Dominant Complex Poles

Illustration

Gap 12

Design Problem

- Shape the root locus plot to achieve the desired pole locations

Root Locus Design: Controller Choice

Procedure

- Sketch the root locus of the plant transfer function and check if the desired properties (damping, overshoot, oscillations, decay) can be fulfilled
- Add controller poles/zeros to change the root locus plot in order to fulfill the desired properties

Example

Gap 13

Root Locus Design: Example

Computation

Gap 14

Root Locus Design: Example

Computation

Gap 15